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THE RADIATION OF ACOUSTIC WAVES BY A PISTON RADIATOR IN A RIGID SCREEN PARTIALLY COVERING A WAVEGUIDE CROSS-SECTION†

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The steady excitation by a piston radiator of normal modes in a plane acoustic waveguide with absolutely rigid side walls is investigated. The radiator, which has mass, is elastically coupled to a rigid screen and is placed at its centre, and the screen is situated at the centre of an absolutely soft end-wall of a semi-infinite waveguide. The Wiener–Hopf–Fock method is used to construct a solution of the problem for the case when the screen half covers the end-wall of the waveguide, while the dimensions of the radiator are arbitrary. The amplitudes of the normal modes and the acoustic fields excited by the radiator in the waveguide are investigated analytically and numerically.

Problems of the diffraction of electromagnetic normal modes by a transverse strip, which is a fixed absolutely soft or rigid screen filling half the cross-section of a waveguide, were solved previously in [1–4].

1. FORMULATION OF THE PROBLEM

The arrangement of the piston radiator in a semi-infinite plane waveguide and the choice of the system of coordinates are shown in Fig. 1. The thick line represents the walls of the waveguide and parts of the end wall, which are a rigid screen S. The horizontal hatched sections represent the absolutely soft part of the end wall of the waveguide. The piston radiator P is represented by the inclined-hatched section, which is placed in the rigid screen in the section $x = 0, -h \leqslant y \leqslant h$. The screen occupies the section $x = 0, h \leqslant |y| \leqslant H/2$.

The complex amplitude $P(x, y)$ of the acoustic pressure in an ideal compressible fluid filling the waveguide in the region $0 < x < +\infty, -H < y < H$ satisfies the homogeneous Helmholtz equation

$$(\Delta + k^2)P(x,y) = 0, \quad k = \omega / c \tag{1.1}$$

where k is the wave number, c is the velocity of sound in the acoustic medium and ω is the angular frequency. The factor $\exp(-i\omega t)$, which specifies the harmonic time dependence of the wave and oscillatory processes, is omitted everywhere.

On the side walls of the waveguide, the following homogeneous Neumann boundary condition is satisfied

$$\partial P / \partial y(x, \pm H) = 0 \tag{1.2}$$

Oscillations are excited in the medium by a time-harmonic force with amplitude F, which acts on the piston radiator. We will denote by A the value of the equivalent pressure which exerts this force on the radiator, $A = F/2h$.

The radiator has a mass M and is attached to an elastic spring of stiffness N. The equation of motion of the piston, taking into account the contact with the acoustic medium, can be written in the form

$$\left(N - M\omega^2\right)U = -\int_{-h}^{h} P(0,y)dy + F \tag{1.3}$$

where U is the amplitude of the displacement of the radiator from the equilibrium position.

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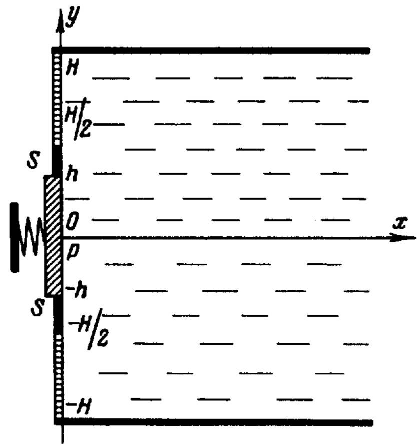

Fig. 1.

The condition for the radiator displacement and the normal component of the displacement vector of the liquid close to it to be equal has the form

$$\rho\omega^2 U = \frac{\partial P}{\partial x}(0,y), \quad -h \leq y \leq h \tag{1.4}$$

where ρ is the density of the acoustic medium.

On the part of the end wall of the waveguide free from the radiator, the homogeneous Neumann boundary condition is satisfied for the rigid fixed screen, while for the soft part of the end wall the homogeneous Dirichlet boundary condition

$$\frac{\partial P}{\partial x}(0,y) = 0, \quad h \leq |y| \leq \frac{H}{2}; \quad P(0,y) = 0, \quad \frac{H}{2} \leq |y| \leq H \tag{1.5}$$

is satisfied.

The acoustic field satisfies the limiting-absorption principle. The Meixner condition is satisfied in the neighbourhood of the point where the piston is joined to the rigid screen, and also at the ends of the rigid screen and the point where the end of the waveguide is joined to its side walls.

2. INTEGRAL EQUATIONS

We will reduce the problem to a system of integral equations.

We will assume that symmetrical normal modes of the following form propagate from the radiator in the waveguide with rigid side walls (taking the symmetry about the waveguide axis into account)

$$Q_n(x,y) = A\sqrt{\frac{k}{\varepsilon_n\gamma_n}}\ \varphi_n(y)\exp(i\gamma_n x), \quad n = 0,1,2,\ldots \tag{2.1}$$

$$\varphi_n(y) = \cos(q_n, y), \quad q_n = \pi n / H, \quad \gamma_n = \sqrt{k^2 - q_n^2}, \quad \varepsilon_0 = 2, \quad \varepsilon_n = 1, \quad n = 1,2\ldots$$

The introduction of the factor A, the value of the equivalent pressure of the active force F acting on the radiator, in the expression for $Q_n(x, y)$ is due to the fact that, in view of the linearity of the problem, the amplitudes of the excited normal modes are proportional to this force.

The normalization of the normal mode is chosen so that all the propagating normal modes $Q_n(x, y)$ ($\operatorname{Im}\gamma_n = 0$ when $\operatorname{Im} k = 0$) transmit through the cross-section of the waveguide the same mean power (per period)

$$W = \frac{1}{2\rho\omega}\ \operatorname{Im}\int_{-H}^{H} \overline{Q_n(x,y)}\ \frac{\partial Q_n(x,y)}{\partial x}\ dy = \frac{A^2 H}{2\rho c} \tag{2.2}$$

Taking into account the symmetry of the problem about the waveguide axis, we have that the acoustic pressures $P(x, y)$ are even with respect to the variable y. Hence, we will seek a solution only in the upper half of the waveguide when $y > 0$, adding along the ray $0 < x < \infty$, $y = 0$ the boundary condition

$$\frac{\partial P}{\partial y}(x,0)=0 \tag{2.3}$$

We will seek the acoustic pressure $P(x, y)$ in the upper half of the waveguide for $x > 0$ in the form of expansions in plane waves

$$P(x,y)=\frac{Ak}{2\pi}\int\limits_{-\infty}^{+\infty} p(\lambda)\ \frac{\operatorname{ch}(\gamma(y-H))}{\gamma\operatorname{sh}(\gamma H/2)}\ \sin(\lambda x)\,d\lambda,\ \ \frac{H}{2}\leqslant y\leqslant H \tag{2.4}$$

$$P(x,y)=-\frac{Ak}{2\pi i}\int\limits_{-\infty}^{+\infty} p(\lambda)\ \frac{\operatorname{ch}(\gamma y)}{\gamma\operatorname{sh}(\gamma H/2)}\ \cos(\lambda x)\,d\lambda+P_*(x,y),\ \ 0<y\frac{H}{2} \tag{2.5}$$

Here $\gamma = \sqrt{(\lambda^2 - k^2)}$, while $P_*(x, y)$ is any particular solution of the Helmholtz equation in the strip $|y| < H/2$ and which satisfies conditions (1.4) and (1.5), but in other respects is fairly arbitrary.

Depending on the chosen boundary conditions on the side walls of the strip for $x > 0$ and $y = \pm H/2$, there will be different particular solutions $P_*(x, y)$. We will choose the solution which satisfies the homogeneous Neumann boundary conditions on these rays. We will seek a particular solution in the form of an expansion in normal modes of a waveguide with rigid walls and width H, i.e. corresponding to the limiting-absorption principle in the form of modes $Q_{2n}(x, y)$, defined by (2.1). In fact, replacing the waveguide width H in (2.1) by $H/2$ is equivalent to doubling the number of normal modes. Hence, we have the following representation (everywhere henceforth summation over n or s is from zero to infinity)

$$P_*(x,y)=u\sum_n\ \alpha_{2n}^*Q_{2n}(x,y),\ \ (u=-i\omega\rho cU/A) \tag{2.6}$$

To find the required amplitudes of the normal modes α_{2n}^* we combine conditions (1.4) and (1.5) in the form

$$\rho\omega^2U\chi_h(y)=\frac{\partial P_*}{\partial x}\ (0,y),\ \ \chi_h(y)=\begin{cases}1, & |y|\leqslant h\\ 0, & h<|y|\leqslant H/2\end{cases} \tag{2.7}$$

This condition now holds for all $|y| < H/2$.

By substituting (2.6) into boundary condition (2.7), after cancelling u we have

$$\chi_h(y)=\sum_n\sqrt{\frac{\gamma_{2n}}{\varepsilon_{2n}k}}\ \ \alpha_{2n}^*\varphi_{2n}(y) \tag{2.8}$$

We multiply both sides of (2.8) by the function $\varphi_{2s}(y)$ ($s = 0, 1, 2, \ldots$) and integrate the expression obtained with respect to the variable y in the section $[-H/2, H/2]$. Taking into account the orthogonality of the functions $\varphi_{2n}(y)$ and $\varphi_{2s}(y)$ in this section when $n \neq s$ we have

$$\alpha_{2n}^*=\sqrt{\frac{k\varepsilon_{2n}}{\gamma_{2n}}}\ \ \eta_{2n};\ \ \varepsilon_0=2,\ \varepsilon_n=1\ \ \text{for}\ \ n=1,2,\ldots \tag{2.9}$$

where η_{2n} are the coefficients of the expansion of the characteristic function $\chi_h(y)$ in a Fourier series in the functions $\varphi_{2n}(y)$

$$\chi_h(y)=\sum_n\ \eta_{2n}\varphi_{2n}(y) \tag{2.10}$$

$$\eta_0=\frac{2h}{H},\ \ \eta_{2n}=\frac{2}{\pi n}\ \sin(q_{2n}h),\ \ n=2,4,\ldots \tag{2.11}$$

Note that in the special case when the radiator completely overlaps the rigid screen ($h = H/2$), the particular solution has the form

$$P_*(x,y)=uQ_0(x,y)=ue^{ikx} \tag{2.12}$$

The required quantities in (2.4) and (2.5) are the function $p(\lambda)$ and the quantity u from (2.6) which is proportional to the particular solution $P_*(x,y)$.

To satisfy the Meixner condition in the neighbourhood of the points of the ends of the rigid screen, we require the following estimate to be satisfied

$$p(\lambda)=O(\lambda^{-\varepsilon}),\ |\lambda|\to\infty,\ \ \varepsilon>0 \tag{2.13}$$

Representations (2.4) and (2.5) for the acoustic pressure $P(x,y)$ ensure that the Helmholtz equation (1.1) and boundary conditions (1.2) and (1.3) are satisfied. These representations also ensure that the conditions on the end wall of the waveguide are satisfied.

In fact, it follows from (2.4) that the function $P(x,y)$, when $H/2<y<H$, is odd with respect to x and, consequently, satisfies condition (1.5). The integral term in (2.5) for the pressure $P(x,y)$ is an even function of the variable x. Hence, the derivative of this term with respect to x when $x=0$ is zero. Conditions (1.4) and (1.5) are satisfied automatically in view of the choice of the form of the particular solution $P_*(x,y)$.

From the continuity of the acoustic pressure in the waveguide and of the y-component of the displacement vector of the acoustic medium along the ray $0<x<\infty$, $y=H/2$ we have the equations

$$P\left(x,\frac{H}{2}-0\right)=P\left(x,\frac{H}{2}+0\right),\ \ \frac{\partial P}{\partial y}\left(x,\frac{H}{2}-0\right)=\frac{\partial P}{\partial y}\left(x,\frac{H}{2}+0\right)$$

Using them, and also representations (2.4) and (2.5), after reduction we obtain two integral equations for finding the required function $p(\lambda)$ when $x>0$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty}p(\lambda)g(\lambda)e^{i\lambda x}d\lambda=P_*\left(x,\frac{H}{2}\right) \tag{2.14}$$

$$g(\lambda)=\frac{k}{\gamma}\operatorname{cth}\frac{\gamma H}{2},\ \ P_*\left(x,\frac{H}{2}\right)=ku\sum_n\frac{(-1)^n}{\gamma_{2n}}\eta_{2n}\exp(i\gamma_{2n}x) \tag{2.15}$$

$$\int_{-\infty}^{+\infty}p(\lambda)\,e^{-i\lambda x}d\lambda=0 \tag{2.16}$$

When obtaining Eq. (2.16) we took into account the fact that on the ray $x>0$, $y=H/2$ the particular solution $P_*(x,y)$, by construction, satisfies the homogeneous Neumann condition, and when obtaining the representation for $P_*(x,H/2)$ we took into account the relations $\varphi_{2n}(H/2)=(-1)^n$ and $\varphi_{2n+1}(H/2)=0$.

3. THE RIEMANN BOUNDARY-VALUE PROBLEM

We will reduce the solution of integral equations (2.14) and (2.16) to the solution of the Riemann boundary-value problem for analytic functions.

By the Wiener–Paley theorem [5] using the relation

$$\exp(i\beta x)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{e^{i\lambda x}}{\lambda-\beta-i0}\,d\lambda,\ \ x>0$$

we conclude from (2.14) and (2.16) that

$$p(\lambda)g(\lambda)=\Phi^+(\lambda)+uf(\lambda),\ \ p(\lambda)=\Phi^-(\lambda) \tag{3.1}$$

$$f(\lambda)=\sum_n f_{2n},\ \ f_{2n}=\frac{k(-1)^n\eta_{2n}}{\gamma_{2n}(\lambda-\gamma_{2n}-i0)}$$

where $\Phi^+(\lambda)$, $\Phi^-(\lambda)$ are analytic functions in the upper half-plane (Im $\lambda > 0$) and the lower half-plane (Im $\lambda < 0$) of the complex variable λ, respectively.

According to the estimate (2.13) we have

$$\Phi^+(\lambda) = O(\lambda^{-1-\varepsilon}), \quad \Phi^-(\lambda) = O(\lambda^{-\varepsilon}), \quad |\lambda| \to \infty \tag{3.2}$$

Eliminating the required function $p(\lambda)$ from (3.1), we arrive at the inhomogeneous Riemann problem [6]

$$\Phi^-(\lambda) g(\lambda) = \Phi^+(\lambda) + uf(\lambda) \tag{3.3}$$

The problem consists of finding two functions $\Phi^+(\lambda)$ and $\Phi^-(\lambda)$ from the linear relation (3.3) that are satisfied on the real axis of the variable λ (Im $\lambda = 0$).

The key fact in solving the Riemann problem is the factorization of the function $g(\lambda)$ from (2.15), i.e. the representation of this function in the form

$$g(\lambda) = g^+(\lambda) g^-(\lambda) \tag{3.4}$$

where $g^+(\lambda)$, $g^-(\lambda)$ are analytic functions in the upper half-plane and lower half-plane of the complex variable λ, respectively, having no zeros in these half-planes. Taking into account the fact that $g(\lambda)$ is a meromorphic function, we obtain the required representation using the theory of infinite products [3, 7, 8]. We have

$$g^+(\lambda) = g^-(-\lambda) = i\sqrt{\operatorname{cth}\frac{kH}{2}} \prod_{n=0}^{\infty} \left(1 + \frac{\lambda}{\gamma_{2n+1}}\right)\left(1 + \frac{\lambda}{\gamma_{2n}}\right)^{-1} \tag{3.5}$$

Here

$$g^{\pm}(\lambda) = O(\lambda^{-1/2}), \quad |\lambda| \to \infty \tag{3.6}$$

Taking representation (3.4) into account we can rewrite Eq. (3.3) in the form

$$\Phi^-(\lambda)\, g^-(\lambda) = \frac{\Phi^+(\lambda)}{g^+(\lambda)} + uF(\lambda) \tag{3.7}$$

We also replace the function $F(\lambda) = f(\lambda)/g^+(\lambda)$ in Eq. (3.7) by the sum of functions that are analytic in the upper half-plane ($F^+(\lambda)$) and lower half-plane ($F^-(\lambda)$) of the variable λ, respectively

$$F(\lambda) = F^+(\lambda) + F^-(\lambda)$$

$$F^+(\lambda) = \sum_n f_{2n}\left(\frac{1}{g^+(\lambda)} - \frac{1}{g^+(\gamma_{2n})}\right), \quad F^-(\lambda) = \sum_n \frac{f_{2n}}{g^+(\gamma_{2n})} \tag{3.8}$$

We convert Eq. (3.7) to the form

$$\Phi^-(\lambda)\, g^-(\lambda) - uF^-(\lambda) = \frac{\Phi^+(\lambda)}{g^+(\lambda)} + uF^+(\lambda) \tag{3.9}$$

By a theorem on analytic continuation through a contour, the left- and right-hand sides of Eq. (3.9) specify a certain unique function $\Phi(\lambda)$, analytic in the whole complex plane of the variable λ. In view of estimates (3.2) and (3.6) this function will be identically equal to zero. Consequently, using the second relation of (3.1) and the second relation of (3.8), we obtain

$$p(\lambda) = \frac{u}{g^-(\lambda)} \sum_n \frac{f_{2n}}{g^+(\gamma_{2n})} \tag{3.10}$$

4. THE ACOUSTIC PRESSURE IN THE WAVEGUIDE

We will obtain a representation of the acoustic pressure in the waveguide in the form of an expansion in normal modes. To do this we will write the integral on the right-hand side of (2.4), taking into account expression (3.10) for the function $p(\lambda)$ in the form

$$P(x,y)=\frac{Au}{2}\sum_n \frac{(-1)^n k\eta_{2n}}{\gamma_{2n}g^+(\gamma_{2n})}\left(I_n^+ + I_n^-\right) \tag{4.1}$$

$$I_n^\pm = \frac{k}{2\pi i}\int_{-\infty}^{+\infty} \frac{\mathrm{ch}(\gamma(y-H))e^{\pm i\lambda x}}{g^-(\lambda)\gamma\,\mathrm{sh}\left(\frac{\gamma H}{2}\right)(\lambda-\gamma_{2n}-i0)}\,d\lambda \tag{4.2}$$

where here we have simultaneously taken either the upper or lower signs.

We will evaluate the integrals for I_n^+ and I_n^- in (4.2) from the theorem of residues, closing the contour of integration by a semicircle of infinite radius in the upper and lower half-planes of the variable λ, respectively, taking (3.1) into account. We will have

$$I_n^+ = (-1)^n g^+(\gamma_{2n})\varphi_{2n}(y)\exp(i\gamma_{2n}x) -$$
$$-\sum_s \frac{2(-1)^{s+1}q_{2s+1}g^+(\gamma_{2s+1})}{(\gamma_{2s+1}-\gamma_{2n})H\gamma_{2s+1}}\varphi_{2s+1}(y)\exp(i\gamma_{2s+1}x)$$
$$I_n^- = \frac{e^{ikx}}{g^+(k)(k+\gamma_{2n})H} + \sum_s \frac{2k(-1)^s}{(\gamma_{2n}+\gamma_{2s})H\gamma_{2s}g^+(\gamma_{2s})}\varphi_{2s}(y)\exp(i\gamma_{2s}x)$$

Using (4.1) we now obtain a representation for the acoustic pressure in the form of the required expansion in normal modes

$$P(x,y)=u\sum_n \alpha_n Q_n(x,y),\quad \alpha_n=\sqrt{\frac{k\varepsilon_n}{\gamma_n}}\,\mu_n \tag{4.3}$$

$$\mu_0=\frac{\eta_0}{2}+\frac{k}{2Hg^+(k)}\sum_s \frac{(-1)^s\eta_{2s}}{(k+\gamma_{2s})\gamma_{2s}g^+(\gamma_{2s})}$$

$$\mu_{2n}=\frac{\eta_{2n}}{2}+\frac{(-1)^n}{g^+(\gamma_{2n})H}\sum_s \frac{(-1)^s\eta_{2s}}{(\gamma_{2n}+\gamma_{2s})\gamma_{2s}g^+(\gamma_{2s})}$$

$$\mu_{2n-1}=(-1)^n q_{2n-1}g^+(\gamma_{2n-1})\sum_s \frac{(-1)^s\eta_{2s}}{(\gamma_{2n-1}-\gamma_{2s})H\gamma_{2s}g^+(\gamma_{2s})},\quad n=1,2,\ldots$$

The coefficients η_{2n} are defined by (2.11).

In the special case when the radiator completely overlaps the rigid screen ($h = H/2$), we have

$$\mu_0=\frac{1}{2}+\frac{1}{4kH(g^+(k))^2}$$

$$\mu_{2n}=\frac{(-1)^n}{g^+(k)g(\gamma_{2n})(\gamma_{2n}+k)H},\quad \mu_{2n-1}=\frac{(-1)^n g^+(\gamma_{2n-1})q_{2n-1}}{g^+(k)(\gamma_{2n-1}-k)kH}\quad n=1,2,\ldots \tag{4.4}$$

We will show that μ_n are the coefficients of the expansion in a Fourier series of the functions $\varphi_n(y)$ of the displacement of the acoustic medium at the end of the waveguide for unit displacement of the piston radiator. In fact, if we calculate the amplitude of the x-component of the displacement vector of the acoustic medium on the end wall of the waveguide $U_1(y)$, we will have, when $|y| < H$, the equation

$$U_1(y)=\frac{1}{\rho\omega^2}\frac{\partial P(0,y)}{\partial x}=U\sum_n \mu_n\varphi_n(y)$$

To determine u in (4.1) we use the equation of motion of the piston (1.3), which gives

$$\frac{N-M\omega^2}{-i\omega\rho c}u=-u\int_{-h}^{h}\sum_{n}\frac{k}{\gamma_n}\alpha_n\varphi_n(y)dy+2h \tag{4.5}$$

From (4.5) we will have

$$u=\frac{\rho c}{Z},\ Z=Z_*+\sum_n Z_n,\ \ Z_*=-i\omega\rho_0\left(1-\frac{\omega_0^2}{\omega^2}\right),\ \ \omega_0=\sqrt{\frac{N}{M}},\ \ \rho_0=\frac{M}{2h} \tag{4.6}$$

$$Z_n=\frac{\rho c\varepsilon_n kH}{2\gamma_n h}\eta_n\mu_n$$

Here Z is the input impedance of the piston radiator in the rigid screen, partially covering the end-wall of the semi-infinite waveguide. This impedance is equal to the ratio of the equivalent pressure A to the value of the complex amplitude of the oscillatory velocity of the radiator $V = -i\omega U$, Z_* is the impedance of the radiator in a vacuum, ω_0 is the natural frequency of the radiator in a vacuum, ρ_0 is the surface density of the radiator and Z_n is the impedance of the interaction between the form of the oscillation of the radiator in the rigid screen and the Nth component of the pressure on the piston $\alpha_n Q_n(0, y)$; it is equal to the ratio of the mean pressure which this component exerts on the radiator to the oscillatory velocity of the radiator V.

If the radiator completely overlaps the rigid screen ($h = H/2$), the equation of motion of the piston can be written in the form

$$\frac{N-M\omega^2}{-i\omega\rho c}u=u\int_{-H/2}^{H/2}\left(\frac{k}{2\pi i}\int_{-\infty}^{+\infty}p(\lambda)\frac{\operatorname{ch}(\gamma y)}{\gamma\operatorname{sh}(\gamma H/2)}\,d\lambda-1\right)dy+H \tag{4.7}$$

We divide the integral on the right-hand side of (4.7) by the sum of the two terms and change the order of integration in the first term. After integration with respect to the variable y we obtain

$$\frac{Z_* u}{\rho c}=\frac{ku}{\pi i}\int_{-\infty}^{+\infty}\frac{d\lambda}{g^+(k)(\lambda-k-i0)g^-(\lambda)(\lambda^2-k^2)}-uH+H$$

We evaluate the integral in the last equation using the theorem of residues, closing the contour of integration in the lower half-plane of the variable λ. The integrand here has a single simple pole at the point $\lambda = -k$, since $g^-(\lambda)$ is an analytic function in this half-plane. After integration we obtain the required quantity u, for which we obtain an expression, similar to (4.6), except that the total impedance Z is calculated from the formula

$$Z=Z_*+2Z_0,\ \ Z_0=\rho c\mu_0 \tag{4.8}$$

where the coefficient μ_0 is defined by the first expression of (4.4).

Comparing (4.8) with the second formula of (4.6), obtained assuming an arbitrary ratio of the dimensions of the piston and the screen, we have the equation

$$Z_0=Z_1+Z_2+\ldots \tag{4.9}$$

which holds when the piston completely overlaps the rigid screen ($h = H/2$).

The final expression for the dimensionless amplitude of the nth normal mode, excited by the piston radiator in the soft end-wall, is obtained from (4.4), taking into account the value of the amplitude of the radiator displacement obtained (4.6). We have

$$\alpha_n=\sqrt{\frac{k\varepsilon_n}{\gamma_n}}\frac{\rho c\mu_n}{Z} \tag{4.10}$$

The acoustic pressure in the waveguide excited by the piston radiator placed in its end wall, according to representation (4.3), taking (4.10) into account, is given by the expression

$$P(x,y) = \frac{Ak\rho c}{Z} \sum_n \frac{\mu_n}{\gamma_n} \varphi_n(y) \exp(i\gamma_n x) \tag{4.11}$$

5. THE RESULTS OF NUMERICAL CALCULATIONS

The results of calculations obtained using (4.10) and (4.11) are shown in Figs 2–4. In all the calculations we assumed that $\rho_0/(\rho H) = M/(2\rho h H) = 1/2$.

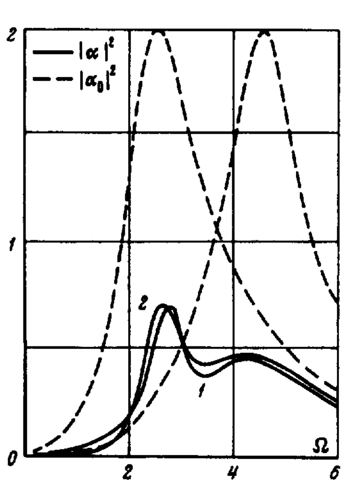

Fig. 2.

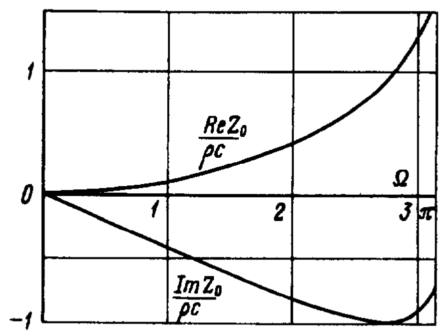

Fig. 3.

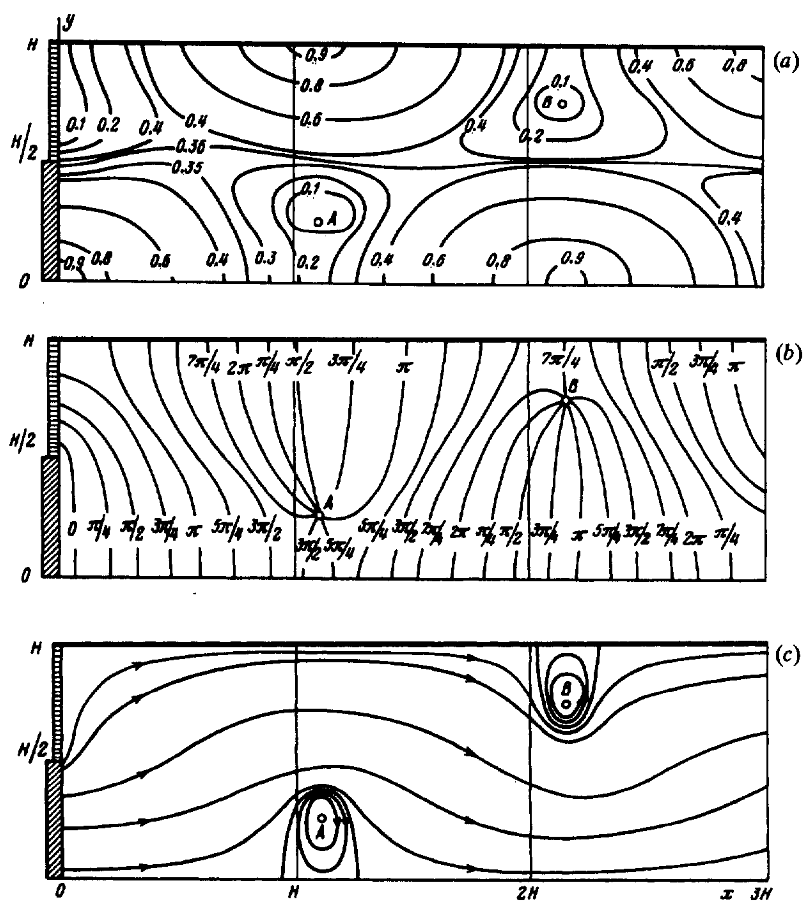

Fig. 4.

Note that the power flux W_n, averaged over a period of the oscillations, carried away from the end wall by the propagating wave with number n, corresponding to (4.11), can be calculated from the formula $W_n = |\alpha_n|^2 W$, where W is given by (2.2). The quantity $|\alpha_n|^2$ will be called the power excitation coefficient of the nth normal mode.

In Fig. 2 we show the power excitation coefficient of the normal mode of the piston type $|\alpha_0|^2$ (the continuous curves) as a function of the dimensionless frequency of the exciting force $\Omega = kH$. The radiator is situated at the centre of the end wall of the waveguide and completely overlaps the rigid screen ($h = H/2$). The dimensionless natural frequency of the piston radiator in a vacuum is $\Omega_* = \omega_0 H/c$, where Ω_1 is the generation frequency of the normal mode $Q_1(x, y)$ ($\Omega_1 = \pi$). Curves 1 correspond to the value $\Omega_* = 2.5 < \Omega_1$ and curves 2 correspond to the value $\Omega_* = 4.5 > \Omega_1$. The dashed curves correspond to the case when the radiator completely overlaps the waveguide cross-section. The parameters of the radiator are chosen so that its impedance in a vacuum is always equal to Z_*

When the radiator completely overlaps the waveguide cross-section, the piston-type wave is the only wave which is excited in the waveguide. The acoustic field in the waveguide has the form $\alpha Q_0(x, y)$, the total impedance of this radiator is calculated from the formula $Z' = Z_* + \rho c$, and the amplitude $\alpha = \sqrt{(2)}\rho c/Z'$.

The graphs shown in Fig. 2 have a resonance form, which is defined by the frequency-dependence of the corresponding total impedances: for $|\alpha_0|^2$ it is Z and for $|\alpha|^2$ it is Z'. Both impedances are complex quantities, the real part of which is the resistive component of the impedance and the imaginary part is the reactive component. The resistive part of the impedance is related to the energy loss in the oscillating mechanical system when it radiates.

When the radiator completely overlaps the waveguide cross-section, the only excited normal mode of the piston type at all frequencies carries away oscillatory energy from the radiator. The interaction impedance of this wave with the form of the radiator oscillation is resistive, equal to ρc, and is independent of the frequency. At the resonant frequency Ω_* the reactive component of the total impedance Z', which here is equal to the impedance of the radiator in a vacuum Z_* vanishes, and $|\alpha|^2$ takes its maximum value, equal to two.

The frequency dependence of the energy excitation coefficient $|\alpha_0|^2$ when the screen partially covers the waveguide cross-section has a more complex form. All normal modes, propagating and non-propagating, are excited in the waveguide. The total impedance of the radiator Z is calculated here using the second formula of (4.6), which, in the special case when $h = H/2$, simplifies and has the form (4.8). All the impedances Z_n, corresponding to the interaction of the radiator and the semi-infinite waveguide, are complex quantities. When $h = H/2$, as was shown earlier in (4.9), the total impedance of the interaction of the radiator and all the normal modes, differing from the piston mode, is equal to the impedance of its interaction with the piston mode Z_0. In Fig. 3 we show the resistive component (Re Z_0, curve 1) and the reactive component (Im Z_0, curve 2) of the impedance Z_0 as a function of the dimensionless frequency Ω for the case when the radiator completely overlaps the rigid screen.

To explain the presence of the reactive part of the impedance Z we will consider the problem of the oscillations of a radiator in a rigid screen, which partially covers the end wall of the semi-infinite waveguide, as the odd part of the problem of an oscillating radiator. The oscillating radiator [9] consists of two in-phase oscillating radiators, arranged symmetrically on both sides of a rigid screen. Discussions similar to those for an oscillating radiator [9] explain the occurrence of a reactive part of the impedances Z_n. In view of the fact that the screen does not completely overlap the waveguide cross-section, it is possible for the medium to transfer from one side of the screen to the other, which leads to levelling the acoustic pressures on the different sides of the screen, and the medium far from it is not compressed. Over a half-period of the oscillations, the pressure levelling is effective at a distance of half a wavelength of the sound from the edge of the radiator. This effect is particularly marked at low frequencies, when the dimensions of the radiator do not exceed the wavelength of the acoustic wave in the medium.

The frequency-dependence of $|\alpha_0|^2$ also has a resonance form, with a shift of the resonance frequency $\Omega' < \Omega_1$ with respect to the frequency Ω_*. The value of Ω' is given by the ratio of the resistive and reactive components of the total impedance Z.

In Fig. 4 we show the results of calculations of the total acoustic field in the waveguide $P(x, y) = |P(x, y)|\exp(i\varphi(x, y))$ (the function $\varphi(x, y)$ describes the phase distribution of the pressure field) when $h = H/2$, $\Omega_* = 4.5$. The frequency of the exciting force was chosen so that $\Omega = 4$.

The distribution of $|P(x, y)/P(0, H/2)|$—the modulus of the normalized pressure (isobars) in the waveguide, is shown in Fig. 4(a). In Fig. 4(b) we show lines of equal phase of the acoustic pressure in the waveguide ($\varphi(x, y) =$ const). The lines of the power flux, averaged over a period, carried away from the radiator, are shown in Fig. 4(c); they are drawn so that the power flux vector, averaged over a period

$$\Pi(x,y) = \frac{1}{2\rho\omega}\operatorname{Im}\left(\overline{P(x,y)}\operatorname{grad}(P(x,y)\right) = \frac{1}{2\rho\omega}|P(x,y)|\operatorname{grad}(\varphi(x,y))$$

at each point of the power streamline is directed along the tangent and is orthogonal to the equiphase lines.

Points A and B in Fig. 4 are points of zero pressure (Fig. 4a), points where the equiphase lines crowd together (Fig. 4b), and points of circulation of the oscillatory energy (Fig. 4c).

Note that the solution obtained is simultaneously of the problem of a piston radiator in a rigid screen filling half the cross-section of an infinite plane waveguide. The radiator is placed at the centre of the screen and excites waves on both sides. The radiated field is odd with respect to the waveguide cross-section, passing through the screen with the radiator. Taking this symmetry into account, we arrive at the problem of a radiator in a semi-infinite waveguide, considered above. In fact, in the cross-section in which the screen is situated, we must, in addition, set up the homogeneous Dirichlet boundary condition.

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